The Pell Equation $x^2 - Dy^2 = 390625$

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ABSTRACT: - Let $D \neq 1$ is a positive non-square integer. In this paper, we obtain some formulas for the integer solutions of the Pell equations $x^2 - Dy^2 = \pm 390625$.

Keywords: Pell's Equation, Solutions of Pell's Equations.s

INTRODUCTION

Pell's equation is a Diophantine equation of a for $x^2 - Dy^2 = 1$, $x, y \in \mathbb{Z}$, where d is a given naturalnumber which is not a square

Pell's equation $x^2 - Dy^2 = 1$ was solved by Lagrange in terms of simple continued fractions. Lagrange was the first to prove that $x^2 - Dy^2 = 1$ has infinitely many solutions in integers if $D \neq 1$ is a fixed positive non-square integer.

The equation $x^2 - Dy^2 = N$, with given integers D and N and unknowns x and$y$, is called Pell's equation. If D is negative, it can have only a finite number of solutions. If D is a perfect square, say $D = a^2$, the equation reduces to $(x - ay)(x + ay) = N$ and again there is only a finite number of solutions. The most interesting case of the equation arises when $D \neq 1$ is a positive non-square.

If the length of the period of $\sqrt{D}$ is 1, all positive solutions are given by $x = P_{2vk-1}$ and $y = Q_{2vk-1}$, if k is odd, and by $x = P_{vk-1}$ and $y = Q_{vk-1}$ if k is even, where $v = 1,2,3,4 \ldots\ldots$ and $\frac{P_n}{Q_n}$ denote the n^{th} convergent of the continued fraction expansion of $\sqrt{D}$. Incidentally, $x = P_{(2v-`1)(k-1)}$ and $y = Q_{(2v-`1)(k-1)}$,

$v = 1,2,3,4 \ldots\ldots$ are the positive solutions of $x^2 - Dy^2 = -1$ provided that 1 is odd.

There is no solution of $x^2 - Dy^2 = \pm 1$ other than (x_v, y_v): $v = 1,2,3 \ldots\ldots$ given by $(x_1 +$ *Dy1v=xv+Dyv* where *x1,y1*the least positive solution is called the fundamental solution.

For completeness we recall that there are many papers in which are considered different types of Pell's Equation. Many authors such as Tekcan[1],[6], Amara Chandoul [2],[3], A. S. Shabani [4], Matthews[5], Kaplan and Wiiliams [7] considered some specific Pell Equations and there integer solutions. A. Tekcan in [1] considered the equation $\boldsymbol{x^2 - Dy^2 = \pm 4}$

And he obtained some formulas for integer solutions. In this paper we extend the work of Amara Chandoul by considering the Pell Equation $\boldsymbol{x^2 - Dy^2 = 390625}$

When $D \neq 1$ be a positive non- square integer.

THE PELL EQUATION $\boldsymbol{x^2 - Dy^2 = 390625}$

THEOREM: 1

Let (x_1, y_1) be the fundamental solution of the Pell equation $x^2 - Dy^2 = 390625$ and let

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1)$$

For $n \geq 1$. Then the integer solutions of the Pell equation $x^2 - Dy^2 = 390625$ are (x_n, y_n)

Where

$$(x_n, y_n) = \left(\frac{u_n}{625^{n-1}}, \frac{v_n}{625^{n-1}}\right) \quad (2)$$

PROOF:

We prove the theorem using the method of mathematical induction.

For n=1, we have from (1)

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$(u_1 \quad v_1) = (x_1 \quad y_1)$$

Which is the a Fundamental Solution of

$x^2 - Dy^2 = 390625$

With the assumption that the Pell equation

$x^2 - Dy^2 = 390625$ is satisfied for (x_{n-1}, y_{n-1});

(i.e.) $x_{n-1}{}^2 - Dy_{n-1}{}^2 = \frac{(u_{n-1})^2 - D(v_{n-1})^2}{(625^{n-2})^2}$

$$x_{n-1}{}^2 - Dy_{n-1}{}^2 = 390625 \quad (3)$$

To prove that the Pell equation

$x^2 - Dy^2 = 390625$ is true for (x_n, y_n).

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} x_1 u_{n-1} + Dy_1 v_{n-1} \\ y_1 u_{n-1} + x_1 v_{n-1} \end{pmatrix}$$

$$u_n = x_1 u_{n-1} + Dy_1 v_{n-1}; \ v_n = y_1 u_{n-1} + x_1 v_{n-1}$$

Hence,

$$x_n{}^2 - Dy_n{}^2 = \frac{(x_1{}^2 - Dy_1{}^2)(u_{n-1}{}^2 - v_{n-1}{}^2)}{(625^{2n-2})}$$

${x_n}^2 - D{y_n}^2 = 390625.$

Hence we conclude that ${x_n}^2 - D{y_n}^2 = 390625.$

Therefore $(x_{n,}y_n)$ is a solution of the Pell equation. Since n is arbitrary, we get all integer solutions of the Pell equation $x^2 - Dy^2 = 390625.$

COROLLARY: 1

Let $(x_{1,}y_1)$ be the fundamental solution of the Pell equation $x^2 - Dy^2 = 390625$, the

$$x_n = \frac{x_1 x_{n-1} + D y_1 y_{n-1}}{625}, \quad y_n = \frac{y_1 x_{n-1} + x_1 y_{n-1}}{625} \qquad (5)$$

And

$$\begin{vmatrix} x_n & x_{n-1} \\ y_n & y_{n-1} \end{vmatrix} = -625 y_1 \qquad (6)$$

PROOF:

We know that

$u_n = x_1 u_{n-1} + D y_1 v_{n-1}$ &

$v_n = y_1 u_{n-1} + x_1 v_{n-1}$

Consequently we get ,

$u_n = x_1 u_{n-1} + D y_1 v_{n-1}$

$$x_n = \frac{[x_1 x_{n-1} + D y_1\, y_{n-1}]}{625} \qquad (7)$$

Similarly,

$v_n = y_1 u_{n-1} + x_1 v_{n-1}$

$$y_n = \frac{[y_1 x_{n-1} + x_1\, y_{n-1}]}{625} \qquad (8)$$

And Hence,

$$\begin{vmatrix} x_n & x_{n-1} \\ y_n & y_{n-1} \end{vmatrix} = x_n y_{n-1} - y_n x_{n-1}$$

$$\begin{vmatrix} x_n & x_{n-1} \\ y_n & y_{n-1} \end{vmatrix} = -625 y_1 .$$

THEOREM: 2

Let $(x_{1,}y_1)$ be the fundamental solution of the Pell equation $x^2 - Dy^2 = 390625$ then $(x_{n,}y_n)$ satisfies the following recurrence relations

$$x_n = \left(\frac{2}{625}x_1 - 1\right)(x_{n-1} + x_{n-2}) - x_{n-3}\,;$$

$$y_n = \left(\frac{2}{625}x_1 - 1\right)(y_{n-1} + y_{n-2}) - y_{n-3}$$

(9)

For $n \geq 4$.

PROOF:

We know that,

$x_n = \frac{[x_1 x_{n-1} + D y_1\, y_{n-1}]}{625}$;

$y_n = \frac{[y_1 x_{n-1} + x_1\, y_{n-1}]}{625}$

The proof will be by induction on n.

Put $n = 2$ in the above equation

$$x_2 = \frac{2}{625}{x_1}^2 - 625 \qquad (10)$$

$$y_2 = \frac{2 y_1 x_1}{625} \qquad (11)$$

Using (5), (10) & (11), we get

Put $n = 3$ in (7) & (8)

$$x_3 = x_1\left(\frac{4}{390625}{x_1}^2 - 3\right) \qquad (12)$$

Similarly

$$y_3 = y_1\left(\frac{4}{390625}{x_1}^2 - 1\right) \qquad (13)$$

Put $n = 4$ in (7) & (8)

$$x_4 = \frac{8}{244140625}{x_1}^4 - \frac{8}{625}{x_1}^2 + 625 \qquad (14)$$

Similarly,

$$y_4 = x_1 y_1\left(\frac{8}{244140625}{x_1}^2 - \frac{4}{625}\right) \qquad (15)$$

Now replacing (10), (11), (12) & (13) in (9)

$$x_4 = \left(\frac{2}{625}x_1 - 1\right)(x_3 + x_2) - x_1$$

$$x_4 = \left(\frac{2}{625}x_1 - 1\right)\left(x_1\left(\frac{4}{390625}{x_1}^2 - 3\right) + \right.$$

2x12−390625625−x1

$$x_4 = \frac{8}{244140625}{x_1}^4 - \frac{8}{625}{x_1}^2 + 625 \qquad (16)$$

$$y_4 = \left(\frac{2}{625}x_1 - 1\right)(y_3 + y_2) - y_1$$

$$y_4 = \left(\frac{2}{625}x_1 - 1\right)\left(\frac{4}{390625}{x_1}^2 y_1 - y_1 + \frac{2}{625}x_1 y_1\right) - y_1$$

$$y_4 = \frac{8}{244140625}{x_1}^3 y_1 - \frac{4}{625}x_1 y_1$$

$y_4 =$
$x_1 y_1 \left(\frac{8}{244140625} {x_1}^2 - \frac{4}{625}\right)$ (17)

Equation (16) & (17) which are the same formula as in (14) & (15).

Therefore (7) holds for $n = 4$.

Now we assume that (7) holds for $n \geq 4$ and we show that it holds for $n + 1$. Indeed by (5) and by hypothesis we have,

Put $n = n + 1$ in (5)

$$x_{n+1} = \frac{x_1 x_n + D y_1 y_n}{625} = \left(\frac{2}{625} x_1 - 1\right)(x_n + x_{n-1}) - x_{n-2}$$

$$y_{n+1} = \frac{x_1 x_n + x_1 y_n}{625} = \left(\frac{2}{625} x_1 - 1\right)(y_n + y_{n-1}) - y_{n-2}$$

Completing the proof.

Then the other solutions are $(x_{2n+1,} y_{2n+1})$
Where $(x_{2n+1,} y_{2n+1})$
$=\left(\frac{u_{2n+1}}{390625^n}, \frac{v_{2n+1}}{390625^n}\right)$ (18) for $n \geq 0$.

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