Study on the solution of linear system of equations and the comparison between iterative methods

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Abstract— In this article we have studied a comparison study on linear system of equations. The Present investigation is intended to study a comparative statement between two methods of finding the matrix inverse. Numerical examples are provided for the methods.

Keywords— Linear system of equations, convergence, iterative methods.

I. INTRODUCTION

Finding the solution is a complicated one in the system of equations. Numerical methods give way to solve complicated problems quickly and easily while compared to analytical solutions. Whether the aim is integration or solution of complex differential equations, there are many tools available to reduce the solution of what can be on occurrence quite difficult methodical arithmetic to simple algebra and some basic loop programming. Solutions of linear simultaneous equations occur quite often in engineering and science. The solution of such equations can be obtained by a numerical method which belongs to one of the two types: Direct or Iterative methods. The direct methods of solving linear equations are known to have their difficulties. System of linear equations arises in a large number of areas both directly in modeling physical situations and indirectly in the numerical solutions of the other mathematical models. In most case it is easier to develop approximate solutions by numerical methods.

In literature, there are many methods to solve the system of equations were studied. There are quite a large number of numerical methods that have been used in the literature to estimate the behavior of system of linear equations. The most important or at least the most used methods are: Motzkin and Schoenberg [7] studied the relaxation method for linear inequalities. The principle of minimize iteration in the solution of the matrix Eigen value problem was derived by Arnoldi [10]. A comparison of direct and indirect solvers for linear system of equations has studied by Noreen Jamil [8]. Kalambi [5] gave a comparison of three iterative methods for the solution of linear equations.

Now, in this article we have compared the solution given in the article[11] with our solution to get a better approximations.

II. BASIC DEFINITIONS AND CONCEPTS

Definition 1:

The unconstrained optimization problems is to *maximize* a real-valued function f of N variable and is defined at a point x^* such that $f(x^*) \le f(x)\ \forall\ x$ near x^*. It is expressed as $\min_x f(x)$ as an objective function and $f(x^*)$ is the minimum value. The local minimum problem is different from the global minimization problem, in which is a *global minimizer*, a point x^* such that $f(x^*) \le f(x)\ \forall\ x$.

Definition 2:

The n-simultaneous system of equations is defined by the following form

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \ldots\ldots\ldots + a_{1n}x_n = s_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \ldots\ldots\ldots + a_{2n}x_n = s_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \ldots\ldots\ldots + a_{3n}x_n = s_3$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 \ldots\ldots\ldots + a_{mn}x_n = s_n$$

where

$a_{11}, a_{12}, a_{13}, \ldots. a_{mn}$, are constant coefficients,

$x_1, x_2, .., x_n$, are the unknowns to be solved,

$s_1, s_2, .., s_n$, are the resultant constants.

The n-simultaneous equations can be expressed in matrix form:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots. & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots. & a_{2n} \\ .. & .. & .. & \ldots. & .. \\ .. & .. & .. & \ldots. & .. \\ a_{m1} & a_{m2} & a_{m3} & \ldots. & a_{mn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{Bmatrix} = \begin{Bmatrix} s_1 \\ s_2 \\ . \\ . \\ s_n \end{Bmatrix}$$

which implies

$$[\mathrm{A}]\{x\} = \{s\}$$

Convergence theorem:

Statement:

If the linear system $[\mathrm{A}]\{x\}=\{s\}$ has a strictly dominant coefficient unit matrix and each equation is solved for its strictly dominant variable, then the iteration will converge to x for any choice of x_0, no matter how errors are arranged.

Proof:

Let $x_1, x_2, .., x_n$, be the exact solution of the system $[\mathrm{A}]\{x\}=\{s\}$. Then

$$\overline{x_i} = \frac{1}{a_{ii}}\left\{b_i - \sum_{j\neq i} a_{ij}\overline{x_j}\right\}$$

$$x_i^{new} = \frac{1}{a_{ii}}\left\{b_i - \sum_{j\neq i} a_{ij}\overline{x_j}\right\}$$

satisfies

$$\varepsilon_i^{new} = \overline{x_i} - x_i^{new}$$

The error of the jth component will be

$$\varepsilon_j^{new} = \overline{x_j} - x_j^{new}$$

By substituting the values in the above equation, we get

$$\varepsilon_i^{new} = \frac{-1}{a_{ii}}\left\{\sum_{j\neq i} a_{ij}\left(\overline{x_j} - x_j\right)\right\}$$

$$= \frac{-1}{a_{ii}}\left\{\sum_{j\neq i} a_{ij}\varepsilon_j\right\}$$

So, if we let $\left|\varepsilon_j\right|_{\max}$ denote the largest $\left|\varepsilon_j\right|$ for $j \neq i$, then we have,

$$\left|\varepsilon_i^{new}\right| = \left|\frac{1}{a_{ii}}\right|\left|\left\{\sum_{j\neq i} a_{ij}\varepsilon_j\right\} \leq \delta\left|\varepsilon_j\right|_{\max}\right|$$

From the above equations it is clear that the error of x_i^{new} is smaller than the error of the other components of x^{new} by a factor of at least δ. The convergence of δ will be therefore assured if δ <1.

Escaletor method:

If the inverse of a matrix A_n of order n is known, then the inverse of a matrix A_{n+1} of order (n+1) can be determined by adding (n+1)th row and (n+1)th column to A_n.

Suppose

$$\mathrm{A} = \begin{bmatrix} \mathrm{A}_1 & : & \mathrm{A}_2 \\ .. & .. & .. \\ \mathrm{A}_3' & : & \alpha \end{bmatrix} \text{ and } \mathrm{A}^{-1} = \begin{bmatrix} X_1 & : & X_2 \\ .. & .. & .. \\ X_3' & : & x \end{bmatrix}$$

where A_2, X_2 are column vectors and A_3', are row vectors and α, x are ordinary numbers.

Also we assume that A^{-1} is known. Actually A_3, X_3' are column vectors since their transposes are row vectors.

Now,

$$\mathrm{AA}^{-1} = I_{n+1} \text{ gives}$$

$$\mathrm{A}_1 X_1 + \mathrm{A}_1 X_3' = I_n \qquad (1)$$

$$\mathrm{A}_2 X_2 + \mathrm{A}_2 x = 0 \qquad (2)$$

$$\mathrm{A}_3' X_1 + \alpha X_3' = 0 \qquad (3)$$

$$\mathrm{A}_3' X_2' + \alpha X = 1 \qquad (4)$$

From (2)

$$X_2 = -\mathrm{A}_1^{-1}\mathrm{A}_2 x \qquad (5)$$

and using this, (4) gives

$$\left(\alpha\text{-}\mathrm{A}_3'\mathrm{A}_1^{-1}\mathrm{A}_2\right)x = 1.$$

Hence x and then X_2 can be found.

Also from (1),

$$X_1 = \mathrm{A}_1^{-1}\left(I_n - \mathrm{A}_2 X_3'\right)$$

and using this, (3) gives

$$\left(\alpha\text{-}\mathrm{A}_3'\mathrm{A}_1^{-1}\mathrm{A}_2\right)X_3' = \mathrm{A}_3'\mathrm{A}_1^{-1}$$

hence X_3' and then X_1 are determined.

Example:

$$\text{Let } \mathrm{A} = \begin{bmatrix} 12 & 13 & 5 & 3 \\ 7 & 0 & 12 & 8 \\ 5 & 6 & 2 & 1 \\ 8 & 4 & 15 & 10 \end{bmatrix}$$

We have

$$A = \begin{bmatrix} 12 & 13 & 5 & : & 3 \\ 7 & 0 & 12 & : & 8 \\ 5 & 6 & 2 & : & 1 \\ .. & .. & .. & .. & .. \\ 8 & 4 & 15 & : & 10 \end{bmatrix} = \begin{bmatrix} A_1 & : & A_2 \\ .. & .. & .. \\ A_3' & : & \alpha \end{bmatrix}$$

so that

$$A_1 = \begin{bmatrix} 12 & 13 & 5 \\ 7 & 0 & 12 \\ 5 & 6 & 2 \end{bmatrix}, A_2 = \begin{bmatrix} 3 \\ 8 \\ 1 \end{bmatrix}$$

$$A_3' = [8 \quad 4 \quad 15] \text{ and } \alpha = 10.$$

We find

$$A^{-1} = \begin{bmatrix} 1.2857 & -0.0714 & -2.7857 \\ -0.8214 & 0.0178 & 1.9464 \\ -0.75 & 0.125 & 1.625 \end{bmatrix}$$

Let $A^{-1} = \begin{bmatrix} X_1 & : & X_2 \\ .. & .. & .. \\ X_3' & : & x \end{bmatrix}$

Then $AA^{-1} = I$.

Hence

$$A_3' A_1^{-1} A_2$$

$$= [8 \quad 4 \quad 15] \begin{bmatrix} 1.2857 & -0.0714 & -2.7857 \\ -0.8214 & 0.0178 & 1.9464 \\ -0.75 & 0.125 & 1.625 \end{bmatrix} \begin{bmatrix} 3 \\ 8 \\ 1 \end{bmatrix}$$

$$= 8.125$$

Therefore,

$$\left(\alpha - A_3' A_1^{-1} A_2\right) x = 1.$$

Becomes

$$(10 - 8.125)\, x = 1$$

$$i.e.,\, x = 0.5333$$

Also,

$$X_2 = -A_1^{-1} A_2 x$$

$$= -\begin{bmatrix} 1.2857 & -0.0714 & -2.7857 \\ -0.8214 & 0.0178 & 1.9464 \\ -0.75 & 0.125 & 1.625 \end{bmatrix} \begin{bmatrix} 3 \\ 8 \\ 1 \end{bmatrix} (0.5333)$$

$$= \begin{bmatrix} 0.2667 \\ -0.2002 \\ 0.1999 \end{bmatrix}$$

Then

$$\left(\alpha - A_3' A_1^{-1} A_2\right) X_3' = A_3' A_1^{-1}$$

becomes

$$(10 - 0.4568)\, X_3' = -[0.5166 \quad 1.9834 \quad 1.2223]$$

$$X_3' = -[0.0145 \quad 1.0123 \quad 1.8765]$$

Finally

$$X_1 = A_1^{-1}\left(I - A_2 X_3'\right)$$

$$= \begin{bmatrix} 2 & 0 & -12 \\ -0.2544 & 1 & 14 \\ 1.2564 & -0.6548 & -0.4789 \end{bmatrix}$$

Hence

$$A^{-1} = \begin{bmatrix} X_1 & : & X_2 \\ .. & .. & .. \\ X_3' & : & x \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & -12 & 0.2667 \\ -0.2544 & 1 & 14 & -0.2002 \\ 1.2564 & -0.6548 & 14 & 0.1999 \\ -0.0145 & 1.0123 & 1.8765 & 0.5333 \end{bmatrix}$$

Iterative method:

Suppose we wish to compute A^{-1} and we know that B is an approximate inverse of A. Then the error matrix is given

$$E = AB - I$$

$$\text{or} \quad AB = I + E$$

$$\text{by i.e., } (AB)^{-1} = (I + E)$$

$$\text{i.e., } B^{-1}A^{-1} = (I + E)^{-1}$$

$$or\ A^{-1} = B(1 - E + E^2 - .....),$$

Provided the series converges. Thus we can find further approximations of A^{-1}, by using

$$A^{-1} = B(1 - E + E^2 - .....)$$

Example:

Now, using the method, we can find the inverse of

$$A = \begin{bmatrix} 2 & 11 & 8 \\ 7 & 0 & 4 \\ 3 & 2 & 2 \end{bmatrix} \text{taking}$$

$$B = \begin{bmatrix} 0.5 & 1.8 & -0.8 \\ 0.25 & 0.25 & -0.14 \\ -2.35 & -4.8 & 1.6 \end{bmatrix}$$

Here

$$E = AB - I$$

$$= \begin{bmatrix} 2 & 11 & 8 \\ 7 & 0 & 4 \\ 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0.5 & 1.8 & -0.8 \\ 0.25 & 0.25 & -0.14 \\ -2.35 & -4.8 & 1.6 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -0.004 & 0 & 0 \\ -0.255 & 0 & 0 \\ -0.07 & 0.01 & -0.06 \end{bmatrix}$$

Therefore,

$$E^2 = \begin{bmatrix} 0.0004 & 0 & 0 \\ 0.0022 & 0 & 0 \\ -0.221 & -0.0047 & -0.0047 \end{bmatrix}$$

To the second approximation, we have

$$A^{-1} = B(1 - E + E^2 - .....)$$

$$= \begin{bmatrix} 0.5689 & 2.4587 & -0.2145 \\ 0.1235 & 0.2581 & -0.0147 \\ -0.8644 & -2.3654 & 3.1458 \end{bmatrix}$$

COMPARISON OF NUMERICAL RESULTS

We have compared the efficiency of the seven numerical methods by taking a 6×6 system of linear equations as follows:-

$$\begin{pmatrix} -4 & -1 & 0 & 0 & -1 & 0 \\ 0 & -4 & 1 & 1 & 0 & 1 \\ 1 & -1 & -4 & -1 & 0 & -1 \\ 0 & -1 & -1 & -4 & 0 & 1 \\ 1 & 1 & 1 & 0 & -4 & -1 \\ 1 & 0 & 0 & 0 & -1 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Results produced by the above diagonally dominant linear system of equations are given in the following table and the graphical solution is given in the figure below.

Methods	**Number of Iterations**	**Computer time in seconds**
Gauss-Seidel	24	1.23
Generalized Gauss-Jacobi	13	0.58
Generalized Gauss Seidel	9	0.32
Iterative	28	2.58
Escletor	-	0.34

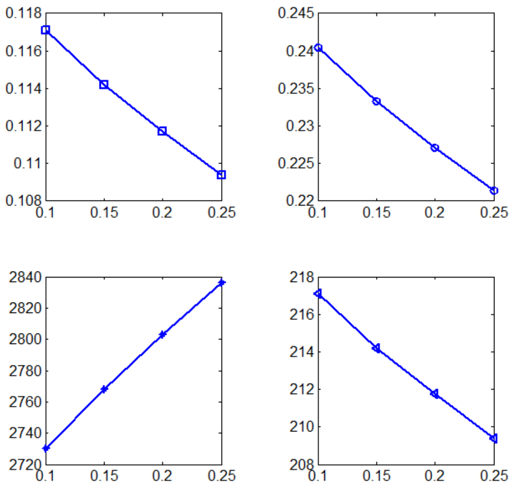

Figure 1 Comparison Results of various methods

III. CONCLUSIONS

Linear systems of equations have a complicated solution. Based on the various systematical comparison we have compared the solution to the methods specified. In General, the errors of estimation or rounding and truncation are introduced when we are using various numerical methods or algorithms and computing. We can find the difference of opinion in linearity and degree. In this paper, we have compared the difference between the solutions when a matrix inversion is to be found by the two methods. The iterative method shows the errors occur in the Escaletor method using an example. In future, it is proposed to study the methods for solving nonlinear equations.

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