Escape Time Fractals of Inverse Tangent Function

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Abstract-The generation of fractals and study of the dynamics of transcendental function is one of emerging and interesting field of research nowadays. We introduce in this paper the complex dynamics of inverse tangent function for $n \geq 2$ and applied Ishikawa iteration to generate new Relative Superior Mandelbrot sets and Relative Superior Julia sets. Our results are entirely different from those existing in the literature of transcendental function.

Keywords- Complex dynamics, Relative Superior Julia set, Relative Superior Mandelbrot set.

I. INTRODUCTION

Fractals are the mathematical fireworks thriving on the new horizons of research in modern analysis and computers. Fractal Theory is an exciting branch of Mathematical Sciences, whose mere existence have worried the founders of modern analysis and soinrecent more sympathetic light has been she don't heseentities. They can be age metrical representation of ubiquitous natural objects like clouds, rivers, and forests. These all are fractals in nature and can be modeled on a computer using a recursive algorithm of computer graphics. They unexpectedly arise in the dynamics of simple dynamical systems. Yet, the usefulness of fractals as a bonafidegeo metrical object has not been fully exploited: algorithmically tools to compute fractals need to be developed; on the contrary real analysis and analytical geometry provide an efficient way to deal with smooth curves and manifolds. The structures like network of veins and shape of mountains are such cases that are hopeless for classical geometry to model them. According to Pickover, the mathematics behind the fractals begantotakeshapeinthe17thcenturywhen the mathematician and philosopher Leibniz pondered recursive self-similarity, although hemade the mistake of thinking that only the straight line was self-similarin this sense. In his writings, Leibniz used the term "fractional exponents", but lamented that "Geometry" did not yet know of them.Indeed, according to various historical accounts, after that point few mathematicians tackled the issues and the work of those who did remained obscured largely because of resistance to such unfamiliar emerging concepts, which were some time referred to as "mathematical monsters".

Thus, it was not until two centuries had passed that in 1872 Karl Weierstrass presented the first definition of a function with agraph that would to day be considered fractal, having the non-in tuitiveproperty of being everywhere continuous but differentiable. Not long after that, in 1883, George Cantor, who attended lectures by Weierstrass, published examples of subsets of their al line known as Cantor sets, this had unusual properties and are now recognized as fractals.AlsointhelastpartofthatcenturyFelixKleinandHenriPoincaréintroduceda category of fractals that is known as "self-inverse fractals". One of the next milest onescamein 1904, when Helge Von Koch, extending ideas of Poinc aré and dissatisfied with Weierstrass's abstract and analytic definition, gavea more geometric definition including hand drawnimages of a similar function, which is now called the Koch curve.

The study of transcendental function has emerged out as discrete dynamical systems in numerical and complex analysis. It forms a rich dynamics for well-known Julia sets and Mandelbrot sets [8]. On the other hand, the dynamics of iterated polynomials are one of the greatest pioneering works of Doaudy and Hubbard [10]. Given a polynomial of degree $n \geq 2$ the most important set is the Julia set J consisting of the points $z \in C$ which have no neighborhood in family of iterates, forms a normal family. Especially for the polynomials, one can start with the set of points I which converge to infinity under iteration (escaping points) and its complement $K = C / I$ is known as filled in Julia sets and it consists of points with bounded orbits. In other words, the Julia set J_c of the function Q_c where $Q_c = z^2 + c$ is either totally disconnected or connected. Its counterpart, Mandelbrot set for a family Q_c is defined as $M_c = \{c \in C : \text{orbit of } 0 \text{ under iteration by } Q_c \text{ is bounded}\}$ For $|c| > 2$, orbit of 0 escapes to ∞ so only $|c| \leq 2$

is considered. For any n, $|Q_c^{(n)}(0)| > 2$, then the orbit of 0 tends to infinity[8].

The dynamics of cosine and sine function as revealed in the past literature states that the points that converge to ∞ under iteration are organized in the form of rays. It is well known that the set of escaping points is an open neighborhood of ∞, which can be parameterized by dynamic rays. As the tangent function is comprised of sine and cosine function, thus it will undertake most of the properties of both the functions. For the entire transcendental functions, the point ∞ is an essential singularity (rather than super attracting point). Ereneko[11] studied that for every entire transcendental functions, the set of escaping points is always non-empty. His query was answered in an affirmative way by R. L. Devaney[5,6 &7], for the special case of Exponential function, where every escaping point can be connected to ∞, along with unique curve running entirely through the escaping points.

This paper studies the dynamical behavior of inverse tangent function also defined as arc tangent function. Fixed points are determined using Relative Superior Ishikawa iterates to develop an entirely new class of fractal images for this transcendental function. Escape criteria of polynomials are used to generate Relative Superior Mandelbrot Sets and Relative Superior Julia Sets. Our results are quite different from existing results in literature as we determined the connectivity of the Julia Sets using Ishikawa iterates.

II. PRELIMINARIES

The process of generating fractal images from $z \to \arctan(z^n)+c$ is similar to the one employed for the self-squared function[17]. Briefly, this process consists of iterating this function up to N times. Starting from a value z_0 we obtain $z_1, z_2, z_3, z_4, \ldots$ by applying the transformation $z \to \arctan(z^n)+c$.

Definition2.1: Ishikawa Iteration [13]: Let X be a subset of real or complex numbers and $f: X \to X$ for $x_0 \in X$, we have the sequences $\{x_n\}$ and $\{y_n\}$ in X in the following manner:

$$y_n = s'_n f(x_n) + (1 - s'_n) x_n$$

$$x_{n+1} = s_n f(y_n) + (1 - s_n) x_n$$

where $0 \le s'_n \le 1$, $0 \le s_n \le 1$ and $\{s'_n\}$ & $\{s_n\}$ are both convergent to non zero number.

Definition 2.2[4, 18]: The sequences $\{x_n\}$ and $\{y_n\}$ constructed above is called Ishikawa sequences of iterations or Relative Superior sequences of iterates. We denote it by $RSO(x_0, s_n, s'_n, t)$. Notice that $RSO(x_0, s_n, s'_n, t)$ with $s'_n = 1$ is $SO(x_0, s_n, t)$ i.e. Mann's orbit and if we place $s_n = s'_n = 1$ then $RSO(x_0, s_n, s'_n, t)$ reduces to $O(x_0, t)$.

We remark that Ishikawa orbit $RSO(x_0, s_n, s'_n, t)$ with $s'_n = 1/2$ is relative superior orbit.

Now we define Mandelbrot sets for function with respect to Ishikawa iterates. We call them as Relative Superior Mandelbrot sets.

Definition 2.3[4, 18]: Relative Superior Mandelbrot set RSM for the function of the form $Q_c(z) = z^n + c$, where n = 1, 2, 3, 4… is defined as the collection of $c \in C$ for which the orbit of 0 is bounded *i.e.* $RSM = \{c \in C : Q_c^k(0) : k = 0, 1, 2 \ldots\}$ is bounded.

In functional dynamics, we have existence of two different types of points. Points that leave the interval after a finite number are in stable set of infinity. Points that never leave the interval after any number of iterations have bounded orbits. So, an orbit is bounded if there exists a positive real number, such that the modulus of every point in the orbit is less than this number.

The collection of points that are bounded, *i.e.* there exists M, such that $|Q^n(z)| \le M$, for all *n*, is called as a prisoner set while the collection of points that are in the stable set of infinity is called the escape set. Hence, the boundary of the prisoner set is simultaneously the boundary of escape set and that is Julia set for Q.

Definition 2.4[4, 18]]: The set of points RSK whose orbits are bounded under relative superior iteration of the function Q (z) is called Relative Superior Julia sets. Relative Superior Julia set of Q is boundary of Julia set RSK

III. GENERATING THE FRACTALS

We have used in this paper escape time criteria of Relative Superior Ishikawa iterates for function $z \to arc\sin(z^n)+c$.

Escape Criterion for Quadratics: Suppose that $|z| > \max\{|c|, 2/s, 2/s'\}$, then $|z_n| > (1+\lambda)^n |z|$ and $|z_n| \to \infty$ as $n \to \infty$.So, $|z| \ge |c|$ & $|z| > 2/s$

as well as $|z| > 2/s'$ shows the escape criteria for quadratics.

Escape Criterion for Cubics: Suppose $|z| > \max\{|b|, (|a| + 2/s)^{1/2}, (|a| + 2/s')^{1/2}\}$ then $|z_n| \to \infty$ as $n \to \infty$. This gives an escape criterion for cubic polynomials

General Escape Criterion: Consider $|z| > \max\{|c|, (2/s)^{1/n}, (2/s')^{1/n}\}$ then $|z_n| \to \infty$ as $n \to \infty$ is the escape criterion. (Escape Criterion derived in [4,18]).

Note that the initial value z_0 should be infinity, since infinity is the critical point of $z \to \tan(z^n) + c$. However instead of starting with z_0 = infinity, it is simpler to start with $z_1 = c$, which yields the same result. (*A critical point of* $z \to F(z) + c$ *is a point where* $F'(z) = 0$ *).*

IV. GEOMETRY OF RELATIVE SUPERIOR MANDELBROT SETS AND RELATIVE SUPERIOR JULIA SETS:

The fractals generated from the equation $z \to \arctan(z^n) + c$ possesses symmetry along the real axis

Relative Superior Mandelbrot Sets:

- In case of quadratic polynomial, the central body is maintaining symmetry along the real axis. Secondary lobes are very small initially for s = 1, s' =1. As the value of the set changes to s =0.6, s' =0.3, the central body gets more unified. As the value of s is still more minimized along with s', the central body is merged into one with none of the secondary lobe.

- In case of Cubic polynomial, the central body is showing bifurcation into two equal parts, each part containing secondary lobes. The symmetry of this body is maintained along both axes. For s =0.6, s' =0.3, the central body becomes more distorted on each side.

- In case of Biquadratic polynomial, the central body is divided into three parts, of which one part possesses a major secondary bulb. The body is maintaining symmetry along the real axis. For s =0.6 s' =0.3, the major secondary lobes disappears.

Relative Superior Julia Sets:

- Relative Superior Julia Sets for the transcendental arc tangent function maintainsits symmetry along real axis, for quadratic polynomial, having (n+1) wings.

- The Relative Superior Julia Sets for Cubic function are symmetrical aboutboth the axes *i.e.* along real and imaginary axes as well as possessesreflectional and rotational symmetry.

- The Relative Superior Julia Sets for Biquadractic function is having symmetry along the real axis having (n+1) wings

V. GENERATION OF RELATIVE SUPERIOR MANDELBROT SETS

A. Mandelbrot Sets of Quadratic function:

Fig1:Relative Superior Mandelbrot Setfor s=s'=1

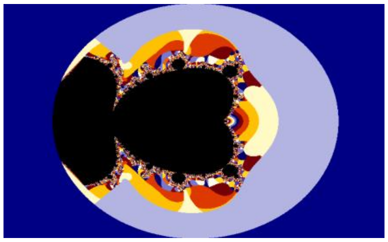


Fig2:Relative Superior Mandelbrot Setfor s=0.6, s'=0.3

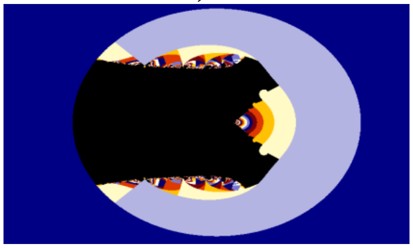
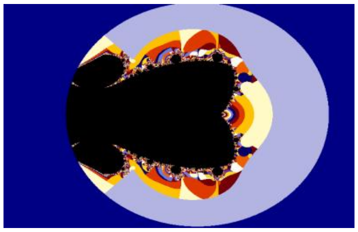


Fig 3:Relative Superior Mandelbrot Setfor s=0.9, s'=0.1



B. Mandelbrot Sets of Cubic function:

Fig1:Relative Superior Mandelbrot Setfor s=s'=1

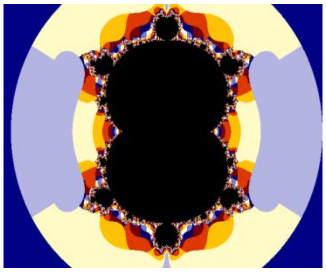


Fig2: Relative Superior Mandelbrot Setfor s=0.6, s'=0.3

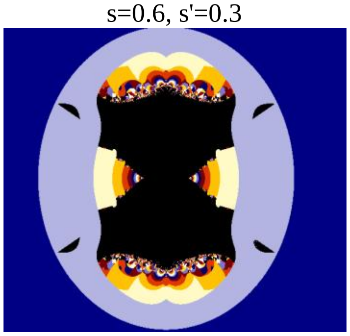
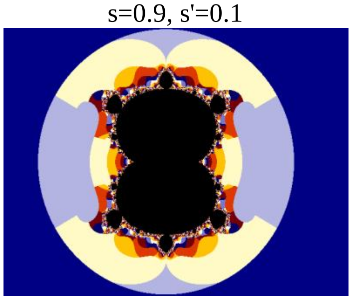


Fig 3:Relative Superior Mandelbrot Setfor s=0.9, s'=0.1



C. Mandelbrot Sets of Biquadratic function:

Fig1:Relative Superior Mandelbrot Setfor s=s'=1

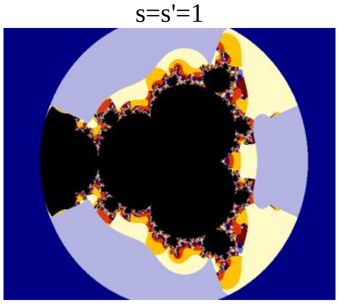


Fig2: Relative Superior Mandelbrot Setfor s=0.6, s'=0.3

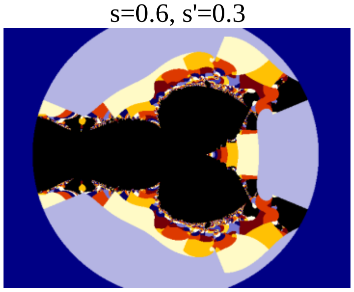
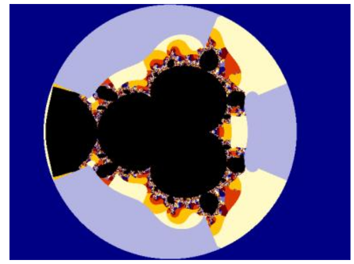


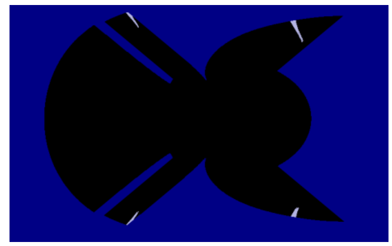
Fig 3: Relative Superior Mandelbrot Set for s=0.9, s'=0.1



VI. GENERATION OF RELATIVE SUPERIOR JULIA SETS

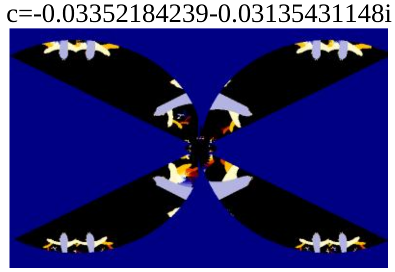
A. Julia sets of Quadratic function:

Fig1:Relative Superior Julia Setfor s=0.6, s'=0.3, c=-0.61455061980+0.00900541716i



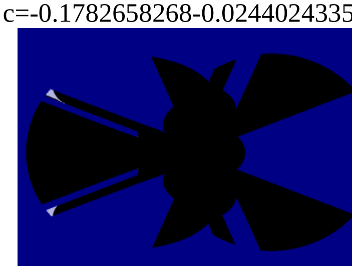
B. Julia Sets of Cubic function:

Fig1:Relative Superior Julia Setfor s=0.9, s'=0.1, c=-0.03352184239-0.03135431148i



C. Julia Sets of Biquadratic function:

Fig1:Relative Superior Julia Setfor s=0.6, s'=0.3, c=-0.1782658268-0.02440243357i



VII. FIXED POINTS:

A. Fixedpointsofquadraticpolynomial

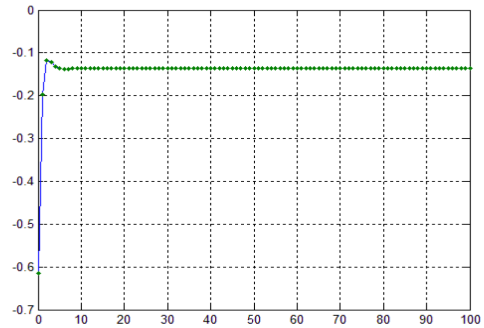
Table 1: Orbit of *F*(*z*) at s=0.6 and s'=0.3 for (z_0= =-0.61455061980+0.00900541716i)

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
1.	0.6146	8.	0.4169
2.	0.3143	9.	0.4168

3.	0.3667	10.	0.4169
4.	0.4072	11.	0.4169
5.	0.4185	12.	0.4169
6.	0.4186	13.	0.4169
7.	0.4174	14.	0.4169

Here we observed that the value converges to a fixed point after 10 iterations.

Figure1. Orbit of $F(z)$ at at s=0.6 and s'=0.3 for (z_0= =-0.61455061980+0.00900541716i)



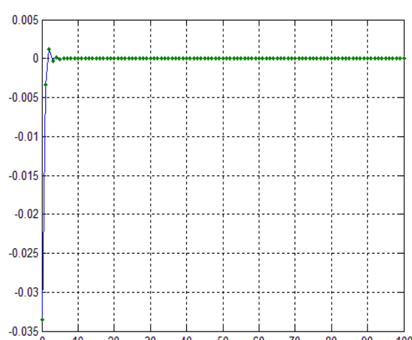
B. Fixedpointsofcubicpolynomial

Table 1 Orbit of F(z) at s=0.9 and s'=0.1 for (z_0=-0.03352184239-0.03135431148*i*)

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
1.	0.0459	6.	0.424
2.	0.4469	7.	0.4237
3.	0.4161	8.	0.4238
4.	0.4261	9.	0.4238
5.	0.423	10.	0.4238

Here we observed that the value converges to a fixed point after 08iterations

Figure 1. Orbit of F(z) at s=0.9 and s'=0.1 for (z_0=-0.03352184239-0.03135431148*i*)



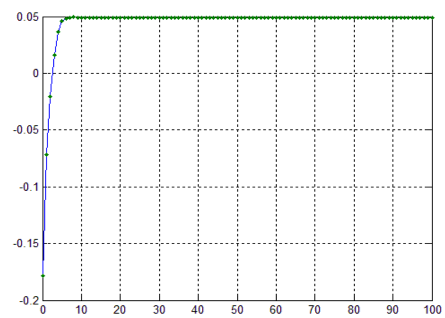
C. FixedpointsofBiquadraticpolynomial

Table 1: Orbit of F(z) at s=0.6 and s'=0.3 for (z_0=-0.1782658268-0.02440243357*i*)

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
1.	0.1799	8.	0.4818
2.	0.2991	9.	0.4814
3.	0.4218	10.	0.4813
4.	0.4698	11.	0.4812
5.	0.4826	12.	0.4812
6.	0.4837	13.	0.4812
7.	0.4826	14.	0.4812

Here we observed that the value converges to a fixed point after 11 iterations

Figure 1. Orbit of F(z) at s=0.6 and s'=0.3 for (z_0=-0.1782658268-0.02440243357*i*)



VIII. CONCLUSION

In this paper we studied the inverse tangent function which is one of the members of transcendental family. Relative Superior Mandelbrot sets possess (n-1)wings, whereas Julia sets possesses (n+1) wing. For even powers, Relative Superior Mandelbrot sets show symmetry only along the real axis while on the other hand, for odd terms, body maintains its symmetry along both axes. The results thus obtained are innovative. Our study is unique in sense that we have used escape time criteria for transcendental function to generate fractals using Relative Superior Ishikawa iterates, otherwise results according to past literature would have shown Julia sets to be disconnected.

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